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Bayesian Estimation of Change Point for Traffic Intensity in M/M/1 Queueing Model under Different Loss Function by Using Quasi Prior

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Abstract: The paper is concerned with the problem of change point for the interarrival time distribution for the $M/M/1$ queueing system. Bayesian estimators of traffic intensities, before the change ρ and after the change ρ_1 and the change point τ are derived. Finally a numerical example is given to illustrate the results.

Keywords: Quasi prior, change point, entropy loss functions, precautionary loss function, squared error loss function.

1. Introduction

The problem of statistical analysis of change point detection, estimation and hypothesis testing concerning the queueing parameters such as the arrival rate, service rate, traffic intensity etc., plays an extremely important role in the decision making analysis of queues. Many authors have studied the parameter estimation problem in queueing models. Bhat and Rao [9] have provided an exhaustive survey of results on inference for queueing systems. The work on various aspects of inference related to queues can be found in Clarke [14], Benes [8], Thiagarajan and Harris [27]. In most of the above works, the estimation and test of hypothesis are confined to the arrival/service rates of Poisson/exponential distribution and birth and death types of queue.

In many real life problems, theoretical or empirical deliberations suggest occasionally changing models. In such models, the main parameter of interest is the change point, which indicates when or where change occurred. There are two fundamental problems of interest related to this parameter, viz., detection of a change and estimation.

A sequence random variables $X_1, X_2, ..., X_n$ is said to have a change-point at τ if $X_i : F_1(x|\theta_1)$ ($i = 1,2,...,\tau$) and $X_i : F_2(x|\theta_2)$ $(i = \tau+1, \tau+2, ..., n)$, where $F_1(x|\theta_1) \neq F_2(x|\theta_2)$. We shall consider the situation in which F_1 and F_2 have known functional forms, but the change point, τ, is unknown. Given a sequence of observations $x_1, x_2, ..., x_n$, the problem we shall be concerned with is that of making inferences about τ. The

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parameters θ_1 and θ_2, which could be vector-valued, may be known or unknown; in the latter case, it might be of the interest to make inferences about these also.

The problem of testing and estimating change points in queueing theory has attracted much attention in the literature. Jain [17] has studied the change point problem for traffic intensity for $M/M/1$ queue in case of changing arrival rate. Acharya and Villarreal [1] have studied change point estimation of traffic intensity for changing service rate for $M/M/1/m$ queueing system.

The problem of estimating change point of the inter-arrival time distribution in the queueing is of great interest. Besides maximum likelihood and least square estimates, the Bayesian method is also a very useful technique for estimating parameters. For Bayesian estimation of queueing parameters we can cite Muddapur [22], McGrath *et al.* [20], McGrath and Singpurwalla [21], Thirvaiyaru and Basawa [28], Armero [3, 4], Armero and Bayarri [5, 6], Ren and Wang [25]. In all these articles, independent random variables like number of arrivals, number of service completion, initial queue length, interarrival times, service times were observed and Beta and Gamma as prior distributions have been used to estimate traffic intensity, arrival and service rates in a single queue. Chowdhury and Mukherjee [13] obtained MLE as well as Bayesian estimator of traffic intensity (ρ) on the basis of queue size at each departure point in $M/M/1$ queue and Cruz *et al.* [15] extended the work for $M/M/s$ queue. Almeida and Cruz [2] obtained better Bayesian estimates for the traffic intensity of $M/M/1$ queue and proposed Jeffreys prior to obtain the posterior and predictive distribution of parameter.

The Bayesian inferential applications can play an important role in study of such problem of change points. In general, an investigator first performs a test to detect a change and, if it is indicated, then the change point is estimated under a specified loss function. Chernoff and Zacks [12], Broemeling [10], Smith [26], Guttman and Menzefricke [16], Raftery and Akman [24], Barry and Hartigan [7] and Lee [19] have studied the change point problem using the Bayesian method. Jain [17] obtained the Bayesian estimator of the change point for the interarrival time distribution in $E_k/G/c$ queueing model under squared error loss function. However, not much literature is found on Bayesian change point estimation in queue.

In this paper we study the change point problem for the traffic intensity for $M/M/1$ queue for changing arrival rate. And same can be done for the service rate also. The preliminaries are given in Section 2. Section 3 deals with the Bayesian estimators of change point τ, ρ and ρ_1 under quasi prior for symmetric and asymmetric loss functions, viz. squared error loss function, precautionary loss function and general entropy loss function respectively. We have given a numerical example in Section 4 to illustrate the results.

2. Preliminaries

Let us consider a $M/M/1$ queueing system with the mean arrival rate λ and mean service time $1/\mu$. Let X denote the number of customers in the system. So under steady state X has the probability mass function

$$p(x \mid \rho) = (1-\rho)\rho^{x}, \quad x = 0,1,2,....; \ \rho = \frac{\lambda}{\mu}. \tag{1}$$

The likelihood function corresponding to (1) is given by

$$L(\rho) = (1-\rho)^{n} \rho^{\sum_{i=1}^{n} x_i} . \tag{2}$$

In the Bayesian approach, we further assume some prior knowledge about the queueing parameter ρ is available to the investigator from past experiences with the underlying queueing system. The prior knowledge can often be summarized in terms of the so-called prior densities on the parameter space of ρ.

Quasi-prior

For the situation where the experimenter has no prior information about the parameter ρ, one may use the quasi density as given by

$$\pi(\rho, c) \propto \frac{1}{\rho^{c}}; \ \rho > 0, \ c > 0. \tag{3}$$

Hence, $c = 0$ leads to a diffuse prior and $c = 1$ to a non-informative prior.

Change-point problem

Let us consider the case where the interarrival time distribution $A(t)$ is assumed to change after some unknown τ, where $1 \le \tau \le n$. Thus

$$A_i(t) = \begin{cases} 1-\exp(-\lambda t), & \text{if } i = 0,1,2,...,\tau \\ 1-\exp(-\lambda_1 t), & \text{if } i = \tau+1, \tau+2, ..., n. \end{cases} \tag{4}$$

Let $(x_1, x_2, ..., x_\tau, x_{\tau+1} ..., x_n)$ be the number of customers present in the system for the time points $t_1, t_2, ..., t_n$. So, for $x_1, x_2, ..., x_\tau$, (1) can be written as

$$p(x \mid \rho) = (1-\rho)\rho^{x_i}; \ i = 1,2,...,\tau, \ \rho = \frac{\lambda}{\mu}$$

and, for $x_{\tau+1}, ..., x_n$

$$p(x \mid \rho_1) = (1-\rho_1)\rho_1^{x_i}; \ i = 1,2,...,\tau, \ \rho_1 = \frac{\lambda_1}{\mu}.$$

Then, the likelihood function in (2) can be written as

$$L(\tau,\rho,\rho_1)=(1-\rho)^{\tau}\rho^{\sum_{i=1}^{\tau}x_i}(1-\rho_1)^{n-\tau}\rho_1^{\sum_{i=\tau+1}^{n}x_i}. \tag{5}$$

3. Bayesian Estimation

Let τ, ρ and ρ_1 are assumed to have independent priors of the following form:

$$\pi(\tau)=\frac{1}{n},\ 1\le\tau\le n \quad \text{(cf. Smith [26]);}$$
$$\pi(\rho)\propto\frac{1}{\rho^{c}},\ 0<\rho<1; \tag{6}$$
$$\pi(\rho_1)\propto\frac{1}{\rho_1^{c_1}},\ 0<\rho_1<1$$

where $c,c_1>0$.

The joint posterior density of τ, ρ and ρ_1 is given by

$$\pi(\tau,\rho,\rho_1)=k[\prod_{i=1}^{\tau}p(x,\rho)\pi(\rho)\prod_{i=\tau+1}^{n}p(x,\rho_1)\pi(\rho_1)]$$
$$=k(1-\rho)^{\tau}\rho^{\sum_{i=1}^{\tau}x_i-c}(1-\rho_1)^{n-\tau}\rho_1^{\sum_{i=\tau+1}^{n}x_i-c_1},$$

where k is a constant such that

$$\sum_{\tau=1}^{n}\int_0^1\int_0^1\pi(\tau,\rho,\rho_1)d\rho d\rho_1=1.$$

$$\Rightarrow\quad \frac{1}{k}=\sum_{\tau=1}^{n}\int_0^1\int_0^1(1-\rho)^{\tau}\rho^{\sum_{i=1}^{\tau}x_i-c}(1-\rho_1)^{n-\tau}\rho_1^{\sum_{i=\tau+1}^{n}x_i-c_1}d\rho d\rho_1$$
$$=\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}.$$

So,

$$\pi(\tau,\rho,\rho_1)=\frac{(1-\rho)^{\tau}\rho^{\sum_{i=1}^{\tau}x_i-c}(1-\rho_1)^{n-\tau}\rho_1^{\sum_{i=\tau+1}^{n}x_i-c_1}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}, \tag{7}$$

where $S_\tau=\sum_{i=1}^{\tau}x_i$, $S_{n-\tau}=\sum_{i=\tau+1}^{n}x_i$

Therefore, the marginal posterior densities of τ, ρ and ρ_1 are computed as:

$$\pi(\tau \mid x)=\int_0^1\int_0^1 \pi(\rho,\rho_1,\tau)\,d\rho\, d\rho_1$$

$$=\frac{\int_0^1\int_0^1 (1-\rho)^{\tau}\rho^{S_\tau-c}(1-\rho_1)^{n-\tau}\rho_1^{S_\tau-c_1}\,d\rho\, d\rho_1}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}$$

$$=\frac{\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}. \qquad (8)$$

$$\pi(\rho \mid x)=\sum_{1}^{n}\int_0^1 \pi(\rho,\rho_1,\tau)\,d\rho_1$$

$$=\frac{\sum_{1}^{n}\int_0^1 (1-\rho)^{\tau}\rho^{S_\tau-c}(1-\rho_1)^{n-\tau}\rho_1^{S_\tau-c_1}\,d\rho_1}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}$$

$$=\frac{\sum_{\tau=1}^{n}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}\rho^{S_\tau-c}(1-\rho)^{\tau}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}. \qquad (9)$$

$$\pi(\rho_1 \mid x)=\sum_{1}^{n}\int_0^1 \pi(\rho,\rho_1,\tau)\,d\rho$$

$$=\frac{\sum_{1}^{n}\int_0^1 (1-\rho)^{\tau}\rho^{S_\tau-c}(1-\rho_1)^{n-\tau}\rho_1^{S_\tau-c_1}\,d\rho}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}$$

$$=\frac{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\rho_1^{S_{n-\tau}-c_1}(1-\rho_1)^{n-\tau}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}. \qquad (10)$$

3.1. Squared error loss function (SELF)

Let us consider the widely used squared error loss function (SELF) which is symmetric and is given by

$$L_1(\hat{\theta}_B)=(\hat{\theta}_B-\theta)^2, \qquad (11)$$

where θ and $\hat{\theta}_B$ are parameter and estimator respectively. Minimizing $E(L_1(\hat{\theta}_B))$, i.e. solving $\frac{dE(L_1(\hat{\theta}_B))}{d\theta}=0$, we get

$$\hat{\theta}_B = E(\theta \mid x). \tag{12}$$

Under the squared error loss function, the Bays estimators of τ, ρ and ρ_1 are obtained as follows:

$$\hat{\tau}_{BS} = E(\tau \mid x)$$

$$= \frac{\sum_{\tau=1}^{n} \tau \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}. \tag{13}$$

$$\hat{\rho}_{BS} = E(\rho \mid x)$$

$$= \frac{\sum_{1}^{n} \int_0^1 (1-\rho)^\tau \rho^{S_\tau - c+1} (1-\rho_1)^{n-\tau} \rho_1^{S_\tau - c_1} d\rho_1}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}$$

$$= \frac{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+2)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+3)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}. \tag{14}$$

Similarly,

$$\hat{\rho}_{1BS} = \frac{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+2)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+3)}}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c+1)\Gamma(\tau+1)}{\Gamma(S_\tau - c+\tau+2)} \frac{\Gamma(S_{n-\tau} - c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1+n-\tau+2)}}. \tag{15}$$

3.2. Precautionary loss function (PLF)

Norstrom [23] introduced an alternative asymmetric loss function and also presented a general class of precautionary loss function as a special case. These loss functions approach infinity near the origin to prevent the underestimation and thus giving conservative estimators, especially when low arrival rates are being estimated which may lead to serious consequences. A very useful and simple asymmetric precautionary loss function (PLF) is given by

$$L_2(\hat{\theta}_B)=\frac{(\hat{\theta}_B-\theta)^2}{\hat{\theta}_B}. \tag{16}$$

Minimizing $E(L_2(\hat{\theta}_B))$ and solving $\frac{dE(L_2(\hat{\theta}_B))}{d\theta}=0,$ we get

$$\hat{\theta}_B=[E(\theta^2\mid x)]^{\frac{1}{2}}. \tag{17}$$

Under PLF, the Bays estimators of τ, ρ and ρ_1 are obtained as follows:

$$\hat{\tau}_{BP}=[E(\tau^2\mid x)]^{\frac{1}{2}}$$

$$=\left[\frac{\sum_{\tau=1}^{n}\tau^2\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}\right]^{\frac{1}{2}}. \tag{18}$$

$$\hat{\rho}_{BP}=[E(\rho^2\mid x)]^{\frac{1}{2}}$$

$$=\left[\frac{\sum_{1}^{n}\int_0^1(1-\rho)^{\tau}\rho^{S_\tau-c+2}(1-\rho_1)^{n-\tau}\rho_1^{S_\tau-c_1}d\rho_1}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+3)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+3)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}\right]^{\frac{1}{2}}. \tag{19}$$

Similarly,

$$\hat{\rho}_{1BP}=\left[\frac{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+3)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+4)}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}\right]^{\frac{1}{2}}. \tag{20}$$

3.3. General entropy loss function (GELF)

Sometimes, the use of symmetric loss function namely squared error loss function (SELF), was found inappropriate. Thus, large attention has been given to asymmetric loss function recently. Calabria and Pulcini [11] proposed a general entropy loss function defined by

$$L_3(\hat{\theta}_B) = (\frac{\hat{\theta}_B}{\theta})^q - qln(\frac{\hat{\theta}_B}{\theta}) - 1;\ q \neq 0. \tag{21}$$

This loss function is a generalization of the Entropy loss function used by many authors when the shape parameter q is taken equal to 1. It may be noted that when $q > 0$, a positive error $(\hat{\theta}_B > \theta)$ causes more serious consequences than a negative error, and vice versa.

Minimizing $E(L_3(\hat{\theta}_B))$ and solving $\frac{dE(L_3(\hat{\theta}_B))}{d\theta} = 0$, we get

$$\hat{\theta}_B = [E(\theta^{-q} \mid x)]^{-\frac{1}{q}}, \tag{22}$$

provided that $E(\theta^{-q} \mid x)$ exists and finite. It can be shown that, when $q = -1$, the Bayes estimator (22) coincides with the Bayes estimator under the squared error loss function. Similarly, when $q = -2$ the Bayes estimator (22) coincides with the Bayes estimator under precautionary loss function.

Under GELF, the Bayes estimator of τ, ρ and ρ_1 are obtained as follows:

$$\hat{\tau}_{BE} = [E(\tau^{-q} \mid x)]^{-\frac{1}{q}}$$

$$= \left[\frac{\sum_{\tau=1}^{n} \tau^{-q} \frac{\Gamma(S_\tau - c + 1)\Gamma(\tau+1)}{\Gamma(S_\tau - c + \tau + 2)} \frac{\Gamma(S_{n-\tau} - c_1 + 1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1 + n - \tau + 2)}}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c + 1)\Gamma(\tau+1)}{\Gamma(S_\tau - c + \tau + 2)} \frac{\Gamma(S_{n-\tau} - c_1 + 1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1 + n - \tau + 2)}} \right]^{-\frac{1}{q}}. \tag{23}$$

$$\hat{\rho}_{BE} = [E(\rho^{-q} \mid x)]^{-\frac{1}{q}}$$

$$= \left[\frac{\sum_{1}^{n} \int_0^1 (1-\rho)^\tau \rho^{S_\tau - c - q} (1-\rho_1)^{n-\tau} \rho_1^{S_\tau - c_1} d\rho}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c + 1)\Gamma(\tau+1)}{\Gamma(S_\tau - c + \tau + 2)} \frac{\Gamma(S_{n-\tau} - c_1 + 1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1 + n - \tau + 2)}} \right]^{-\frac{1}{q}}$$

$$= \left[\frac{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c - q + 1)\Gamma(\tau+1)}{\Gamma(S_\tau - c - q + \tau + 2)} \frac{\Gamma(S_{n-\tau} - c_1 + 1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1 + n - \tau + 2)}}{\sum_{\tau=1}^{n} \frac{\Gamma(S_\tau - c + 1)\Gamma(\tau+1)}{\Gamma(S_\tau - c + \tau + 2)} \frac{\Gamma(S_{n-\tau} - c_1 + 1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau} - c_1 + n - \tau + 2)}} \right]^{-\frac{1}{q}}. \tag{24}$$

Similarly,

$$\hat{\rho}_{1BE}=\left[\frac{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1-q+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1-q+n-\tau+2)}}{\sum_{\tau=1}^{n}\frac{\Gamma(S_\tau-c+1)\Gamma(\tau+1)}{\Gamma(S_\tau-c+\tau+2)}\frac{\Gamma(S_{n-\tau}-c_1+1)\Gamma(n-\tau+1)}{\Gamma(S_{n-\tau}-c_1+n-\tau+2)}}\right]^{-\frac{1}{q}} \quad . \qquad (25)$$

4. Numerical Results

We have generated 2000 random sample of sizes $n=10,20,50$ for given $\lambda=3$, $\lambda_1=4.5$ and $\mu=5$, i.e. $\rho=0.6$ and $\rho_1=0.9$. Since $E(\rho)\geq 0.5$, which we commonly come across in queueing, we have chosen hyper-parameters of quasi prior $c=1.5$ and $c_1=2$. For given change point τ and sample size n, Bayes estimators of change point τ, the parameters before change point (ρ) and after change point (ρ_1) and their risks are computed under different loss functions which are tabulated. From the table it is seen that the estimators are very close to true value as desired.

Table 1. Different Estimates of τ, ρ and ρ_1 and their estimated risks.

Sample size	Estimator	Estimated value	Risk
*n=10, $\tau=5$	τ_{BS}	5.4370	1.2830
	ρ_{BS}	0.6100	0.0984
	ρ_{1BS}	0.9002	0.4967
	τ_{BP}	6.1362	2.8921
	ρ_{BP}	0.6323	0.5037
	ρ_{1BP}	0.9022	0.4967
	$\tau_{BE}(q=-3)$	6.6369	4.3274
	ρ_{BE}	0.6525	3.9563
	ρ_{1BE}	0.8906	1.3043
	$\tau_{BE}(q=3)$	5.9982	1.6779
	ρ_{BE}	0.6199	1.4316
	ρ_{1BE}	0.8902	0.5052
*n=20, $\tau=10$	τ_{BS}	10.7400	1.8920
	ρ_{BS}	0.6094	0.0977
	ρ_{1BS}	0.9022	0.4968
	τ_{BP}	10.1528	2.1918
	ρ_{BP}	0.6002	0.5116

	ρ_{1BP}	0.9251	3.9900
	$\tau_{BE}(q=-3)$	13.1324	6.9999
	ρ_{BE}	0.5906	1.6700
	ρ_{1BE}	0.9245	2.9419
	$\tau_{BE}(q=3)$	12.5766	1.7269
	ρ_{BE}	0.6037	1.4167
	ρ_{1BE}	0.8798	2.4945
*n=50, $\tau=25$	τ_{BS}	24.8170	1.7270
	ρ_{BS}	0.6238	0.1339
	ρ_{1BS}	0.9027	0.6322
	τ_{BP}	24.6457	2.3463
	ρ_{BP}	0.6047	0.6427
	ρ_{1BP}	0.9012	1.7802
	$\tau_{BE}(q=-3)$	23.3495	3.5503
	ρ_{BE}	0.6251	1.7659
	ρ_{1BE}	0.9100	1.6543
	$\tau_{BE}(q=3)$	23.4148	1.7918
	ρ_{BE}	0.6091	1.9302
	ρ_{1BE}	0.9029	1.0952

References

[1] Acharya, S. K., & Villarreal, C. E. (2013). Change point estimation of service rate in $M/M/1$ queue. *International Journal of Mathematics in Opeational Research*, 5(1), 110-120.

[2] Almeida, M. A. C., & Cruz, F. R. B. (2018). A note on bayesian estimation of traffic intensity in single-server markovian queues. *Communications in Statistics – Simulation and Computaion*, 47, 2577-2586.

[3] Armero, C. (1985). Bayesian analysis of $M/M/1/\infty/FIFO$ queues. *Bayesian Statistics 2, eds. J. M.*, 613-617.

[4] Armero, C. (1994). Bayesian inference in Markovian queues. *Queueing Systems*, 15, 419-426.

[5] Armero, C., & Bayarri, M. J. (1994a). Bayesian prediction in $M/M/1$ queues. *Queueing System*, 15, 401-417.

[6] Armero, C., & Bayarri, M. J. (1994b). Prior assessments for prediction in queues. *The statistician*, 43(1) , 139-153.

[7] Barry, D., & Hartigan, J. A. (1993). A Bayesian analysis for change point problems. *Journal of the American Statistical Association*, 88(421), 309-319..

[8] Benes, V. E. (1957). A sufficient set of statistics for simple telephonic exchange model. *The Bell System Technical Journal*, 36, 939-964.

[9] Bhat, U. N., & Rao, S. S. (1987). Statistical analysis of queueing systems. *Queueing System*, 1, 217-247.

[10] Broemeling, L. D. (1972). Bayesian procedure for detecting a change in a sequence of random variables. *Metron*, 30, 214-227.

[11] Calabria, R., & Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. *Microelectronics Reliability*, 34, 789-802.

[12] Chernoff, H., & Zacks, S. (1964). Estimating the current mean of a normal distribution which is subjected to changes in time. *Annals of Mathematical Statistics*, 35, 999-1018.

[13] Chowdhury, S., & Mukherjee, S. P. (2013). Estimation of traffic intensity based on queue length in a single $M/M/1$ queue. *Communications in Statistics – Theory and Methods*, 42(13), 2376-2390.

[14] Clarke, A. B. (1957). Maximum likelihood estimates in a simple queue. *Annals of Mathematical Statistics*, 28, 1036-1040.

[15] Cruz, F. R. B., Quinino, R. C., & Ho, L. L. (2017). Bayesian estimation of traffic intensity based on queue length in a multi-server $M/M/s$ queue. *Communications in Statistics – Simulation and Computation*, 46(9), 7319-7331.

[16] Guttman, I., & Menzefricke, U. (1982). On the use of loss-functions in the change point problem. *Annals of the Institute of Statistical Mathematics*, 34, 319-326.

[17] Jain, S. (1995). Estimating changes in traffic intensity for M/M/1 queueing systems. *Microelectronics Reliability*, 35(11), 1395-1400.

[18] Jain, S. (2001). Estimating the change point of Erlang interarrival time distribution. *INFOR*, 39(2), 200-207.

[19] Lee, C. B. (1998). Bayesian analysis of a change point in exponential families with application. *Computational Statistics and Data Analysis*, 27, 195-208.

[20] McGrath, M. F., Gross, D., & Singpurwalla, N. D. (1987). A subjective Bayesian approach to the theory of queues $I-$Modelling. *Queueing System*, 1, 317-333.

[21] McGrath, M. F., & Singpurwalla, N. D. (1987). A subjective Bayesian approach to the theory of queues $II-$Inference and Information in $M/M/1$ queues. *Queueing System*, 1, 335-353.

[22] Muddapur, M. V. (1972). Bayesian estimation of parameters in some queueing models. *The Institute of Statistical Mathematics*, 24, 327-331.

[23] Norstrom, J. G. (1996). The use of precautionary loss function in risk analysis. *IEEE Transactions on Reliability*, 45(3), 400-403.

[24] Raftery, A. E., & Akman, V. E. (1986). Bayesian analysis of a Poisson process with a change-point. *Biometrika*, 73, 85-89.

[25] Ren, H., & Wang, G. (2011). Bayes estimation of traffic intensity in $M/M/1$ queue under a precautionary function. *Procediea Engineering*, 29, 3646-3650.

[26] Smith, A. F. M. (1975). A Bayesian approach to inference about a change-point in a sequence of random variables. *Biometrika*, 62, 407-416.

[27] Thiagarajan, T. R., & Harris, C. M. (1979). Statistical tests for exponential service from $M/G/1$ waiting time data. *Naval Research Logistic Quarterly*, 26, 511-520.

[28] Thiruvaiyaru, D., & Basawa, I. V. (1992). Empirical Bayes estimation for queueing systems and networks. *Queueing Systems*, 11, 179-202.